Mathematical Proofs

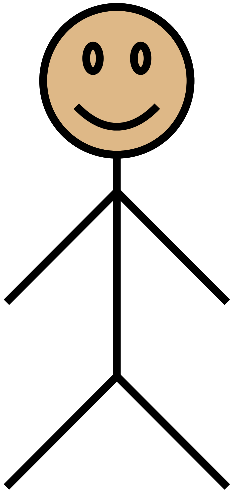
Outline for Today

- ***How to Write a Proof***
  - Synthesizing definitions, intuitions, and conventions.
- ***Proofs on Numbers***
  - Working with odd and even numbers.
- ***Universal and Existential Statements***
  - Two important classes of statements.
- ***Variable Ownership***
  - Who owns what?

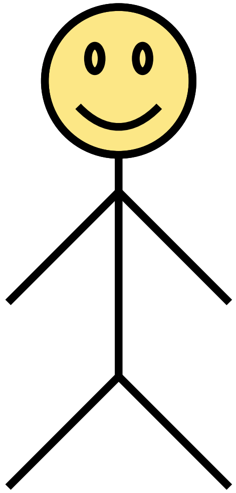
What is a Proof?

Proof as Dialog

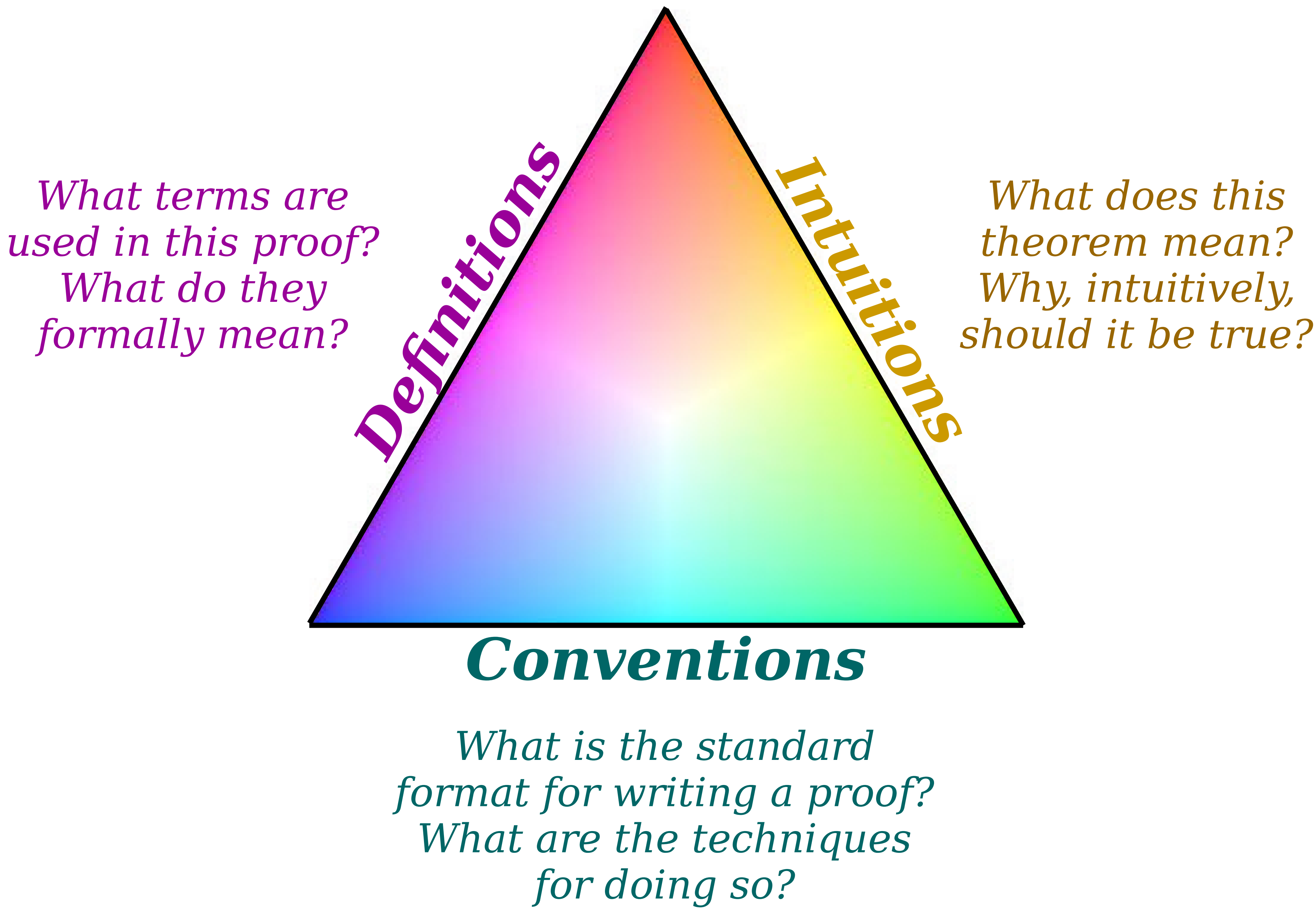
- A mathematical proof is a dialog between two parties: a ***proof writer*** and a ***proof reader***.
  - The ***proof writer*** knows a mathematical fact.
  - The ***proof reader*** is honest but skeptical.
- The proof writer's job is to take the reader on a journey from ignorance to understanding.



Proof Writer (You)

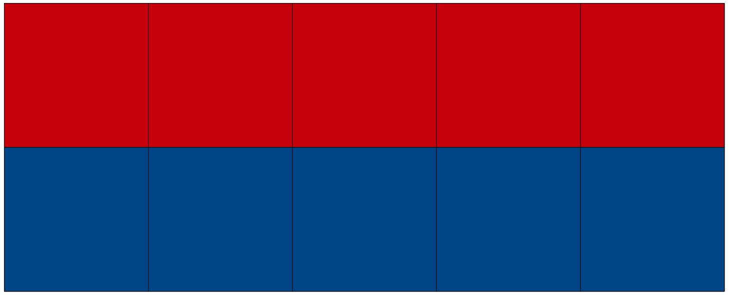


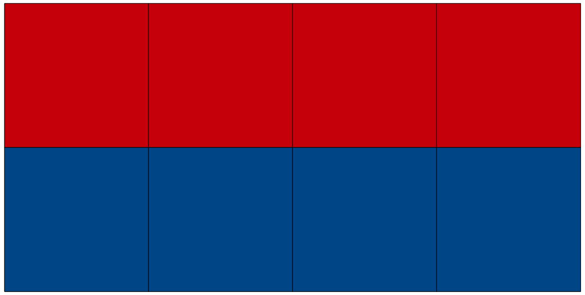
Proof Reader


Definitions
Intuitions
Conventions
What terms are used in this proof? What do they formally mean?
What does this theorem mean? Why, intuitively, should it be true?
What is the standard format for writing a proof? What are the techniques for doing so?

Writing our First Proof

Theorem: If n is an even integer, then n^2 is even.

10  $2 \cdot \mathbf{5}$

8  $2 \cdot \mathbf{4}$

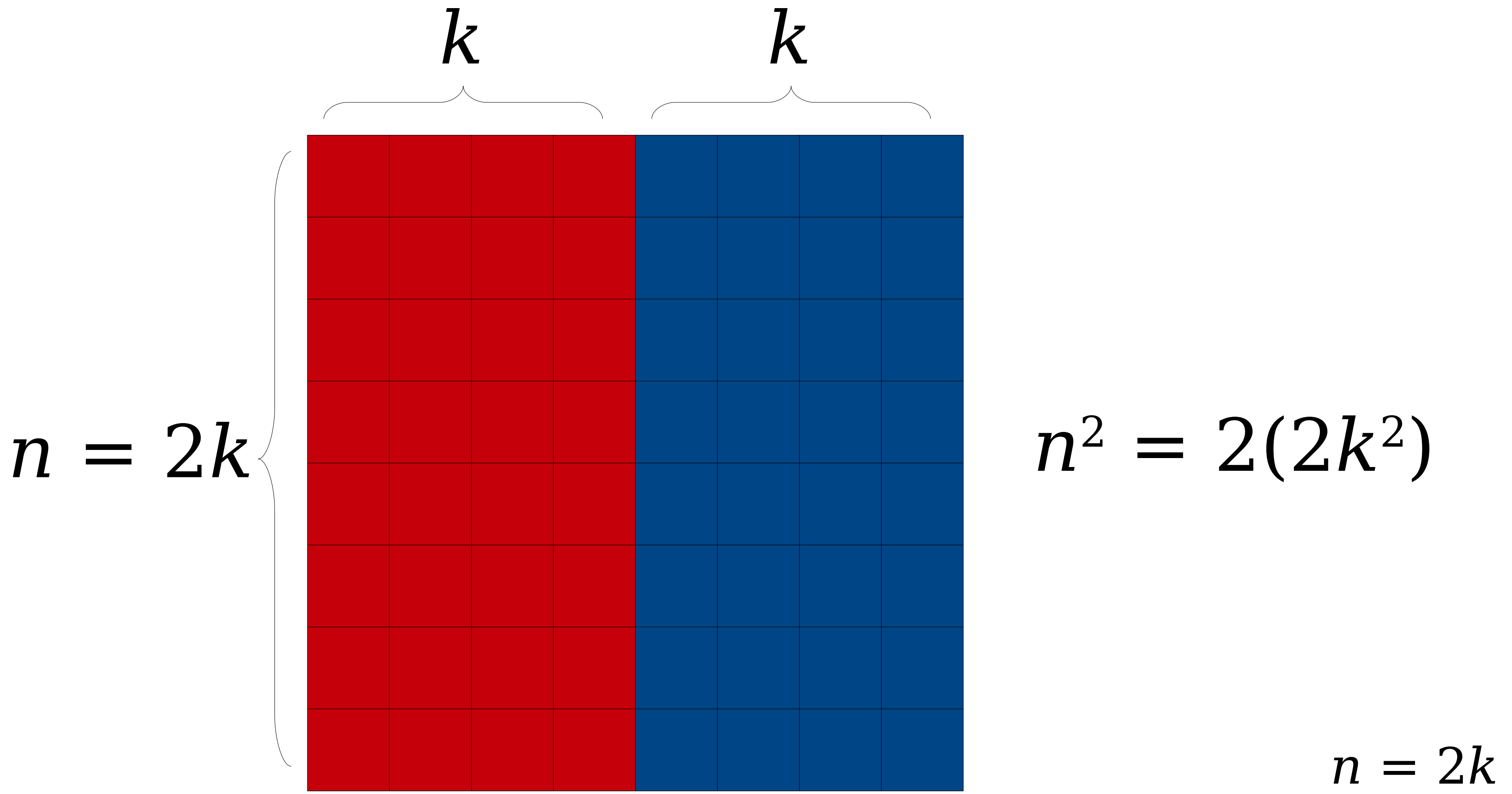
0 $2 \cdot \mathbf{0}$

An integer n is called ***even*** if
there is an integer k where $n = 2k$.

Let's Try Some Examples!

Theorem: If n is an even integer, then n^2 is even.

Let's Draw Some Pictures!



Theorem: If n is an even integer, then n^2 is even.

Our First Proof!

Theorem: If n is an even integer, then n^2 is even.

Proof: Assume n is an even integer. We want to show that n^2 is even.

Since n is even, there is some integer k such that $n = 2k$. This means that

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2). \end{aligned}$$

From this, we see that there is an integer m (namely, $2k^2$) where $n^2 = 2m$. Therefore, n^2 is even, which is what we wanted to show. ■

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This symbol means "end of proof"

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"If *P* is true, then *Q* is true,"

start by asking the reader to assume that ***P*** is true.

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To prove a statement of the form

"If *P* is true, then *Q* is true,"

we assume ***P*** is true, then need to show that ***Q*** is true. Here, we're telling the reader where we're headed.

Our First Proof!

Theorem: If n is an even integer, then n^2 is even.

Proof: Assume n is an even integer. We want to show that n^2 is even.

Since n is even, there is some integer k such that $n = 2k$. This means that

$$n^2 = (2k)^2$$

This is the definition of an even integer. We need to use this definition to make this proof rigorous.

From this, we see that there is an integer m (namely, $2k^2$) where $n^2 = 2m$. Therefore, n^2 is even, which is what we wanted to show. ■

Our First Proof!

Theorem: If

Proof: Assum

show that n

Notice how we use the value of ***k*** that we obtained above. Giving names to quantities, allows us to manipulate them. This is similar to variables in programs.

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Our First Proof!

Theorem: If n is an even integer, then n^2 is even.

Proof: Assume n is an even integer. We want to show that n^2 is even.

Since … h
that n …

Our ultimate goal is to prove that $\boldsymbol{n^2}$ is even. This means that we need to find some $\boldsymbol{m}$ where $\boldsymbol{n^2 = 2m}$. Here, we're explicitly showing how we can do that.

From this, we see that there is an integer m (namely, $2k^2$) where $n^2 = 2m$. Therefore, n^2 is even, which is what we wanted to show. ■

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$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) \end{aligned}$$

Hey, that's what we said we were going to do! We're done.

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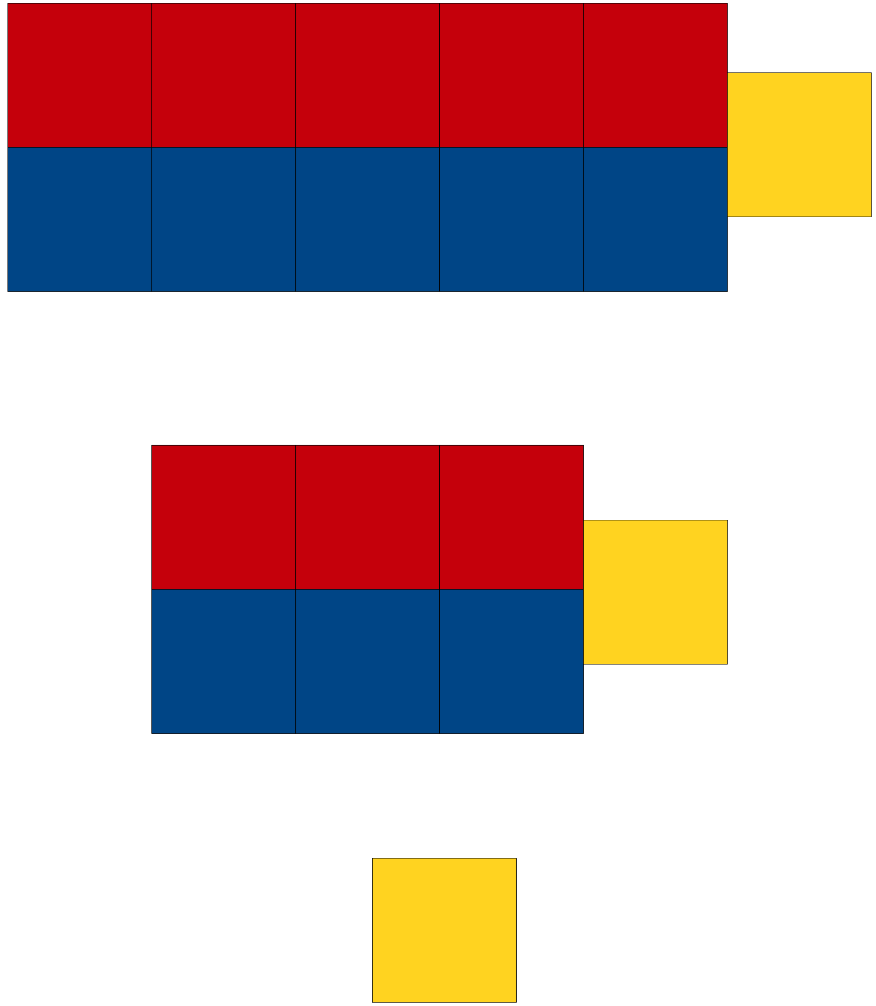
Our Next Proof

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.



11 $\quad 2 \cdot \mathbf{5} + \mathbf{1}$

7 $\quad 2 \cdot \mathbf{3} + \mathbf{1}$

1 $\quad 2 \cdot \mathbf{0} + \mathbf{1}$

An integer n is called ***odd*** if
there is an integer k where $n = 2k+1$.

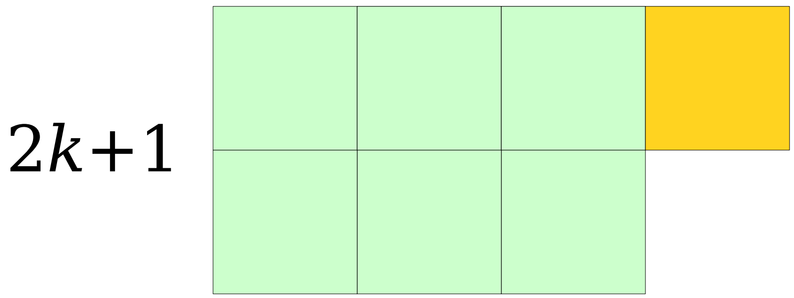
Going forward, we'll assume the following:

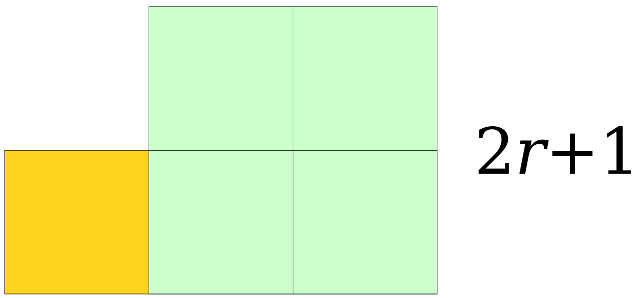
1. Every integer is either even or odd.
2. No integer is both even and odd.

Let's Try Some Examples!

Theorem: For any integers m and n,
if m and n are odd, then $m+n$ is even.

Let’s Do Some Math!





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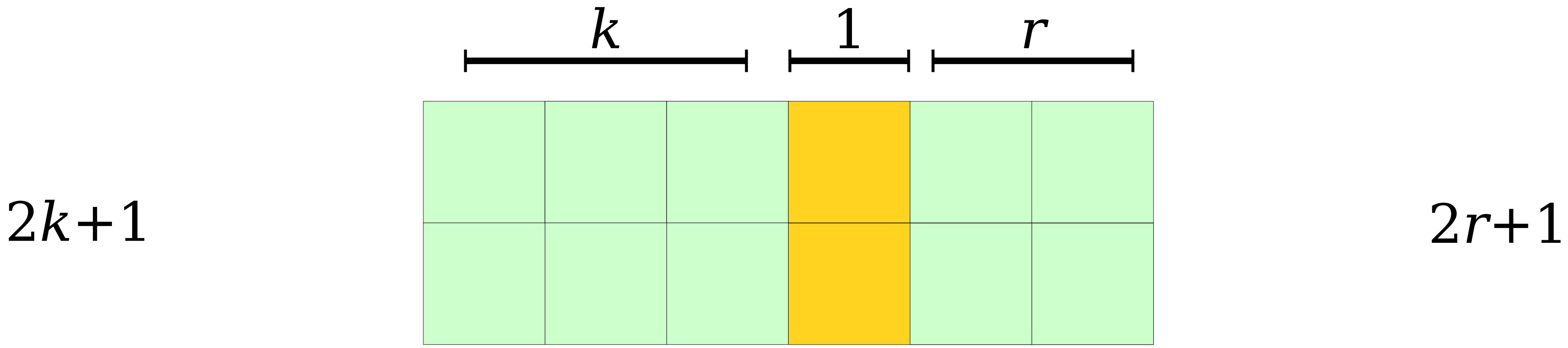

$$(2k+1) + (2r+1) = 2(k + r + 1)$$

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Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Proof: Consider any arbitrary integers m and n where m and n are odd. We need to show that $m + n$ is even.

Since m is odd, we know that there is an integer k where

$$m = 2k + 1. \qquad (1)$$

Similarly, because n is odd there must be some integer r such that

$$n = 2r + 1. \qquad (2)$$

By adding equations (1) and (2) we learn that

$$\begin{aligned} m + n &= 2k + 1 + 2r + 1 \\ &= 2k + 2r + 2 \\ &= 2(k + r + 1). \qquad (3) \end{aligned}$$

Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

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We ask the reader to make an *arbitrary choice*. Rather than specifying what m and n are, we're signaling to the reader that they could, in principle, supply any choices of m and n that they'd like.

By letting the reader pick m and n arbitrarily, anything we prove about m and n will generalize to all possible choices for those values.

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Numbering these equalities lets us refer back to them later on, making the flow of the proof a bit easier to understand.

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This is a complete sentence! Proofs are expected to be written in complete sentences, so you'll often use punctuation at the end of formulas.

We recommend using the "mugga mugga" test – if you read a proof and replace all the mathematical notation with "mugga mugga," what comes back should be a valid sentence.

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Some Little Exercises

- Here's a list of other theorems that are true about odd and even numbers:
  - ***Theorem:*** The sum and difference of any two even numbers is even.
  - ***Theorem:*** The sum and difference of an odd number and an even number is odd.
  - ***Theorem:*** The product of any integer and an even number is even.
  - ***Theorem:*** The product of any two odd numbers is odd.
- Going forward, we'll just take these results for granted. Feel free to use them in the problem sets.
- If you'd like to practice the techniques from today, try your hand at proving these results!

Universal and Existential Statements

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

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This result is true for every possible choice of odd integer n. It'll work for $n = 1$, $n = 137$, $n = 103$, etc.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

We aren't saying this is true for every choice of r and s. Rather, we're saying that *somewhere out there* are choices of r and s where this works.

Universal vs. Existential Statements

- A ***universally-quantified statement*** is a statement of the form

For all *x*, `[some-property]` holds for *x*.

- We've seen how to prove these statements.
- An ***existentially-quantified statement*** is a statement of the form

There is some *x* where `[some-property]` holds for *x*.

- How do you prove an existentially-quantified statement?

Proving an Existential Statement

- Over the course of the quarter, we will see several different ways to prove an existentially-quantified statement of the form

  There is an *x* where `[some-property]` holds for *x*.

- ***Simplest approach:*** Search far and wide, find an *x* that has the right property, then show why your choice is correct.

Let's Try Some Examples!

$$1 = \mathbf{1}^2 - \mathbf{0}^2$$

$$3 = \mathbf{2}^2 - \mathbf{1}^2$$

$$5 = \mathbf{3}^2 - \mathbf{2}^2$$

$$7 = \mathbf{4}^2 - \mathbf{3}^2$$

$$9 = \mathbf{5}^2 - \mathbf{4}^2$$

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Let's Try Some Examples!

$$1 = 2 \cdot \mathbf{0} + 1 = \mathbf{1}^2 - \mathbf{0}^2$$

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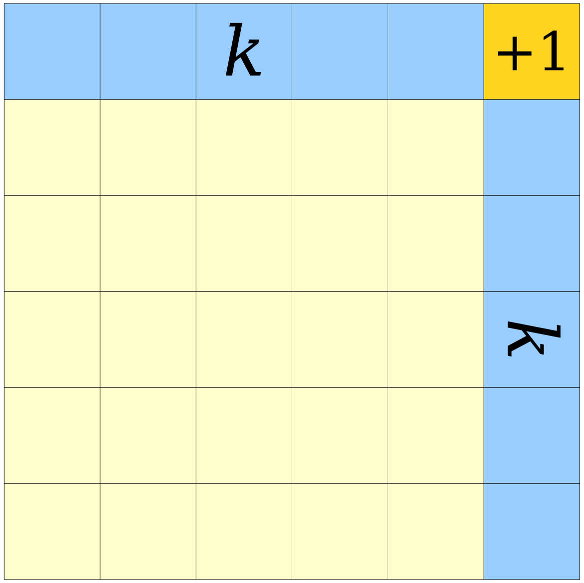
Let's Draw Some Pictures!



$$(k+1)^2 - k^2 = 2k+1$$

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

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Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$
\begin{aligned}
r^2 - s^2 &= (k+1)^2 - k^2 \\
&= k^2 + 2k + 1 - k^2 \\
&= 2k + 1 \\
&= n.
\end{aligned}
$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

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> We ask the reader to make an *arbitrary choice*. Rather than specifying what n is, we're signaling to the reader that they could, in principle, supply any choice n that they'd like.

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$$r^2 - s^2 = (k+ \ldots$$

$$= k^2 + 2k + 1 - k^2$$

$$= 2k + 1$$

$$= n.$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

As always, it's helpful to write out what we need to demonstrate with the rest of the proof.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$r^2 - s^2 = (k+1)^2 \ldots$$

$$= k^2 + 2\ldots$$

$$= 2k + 1\ldots$$

$$= n.$$

This means that $r^2 - s^2 = n$, w… to show. ■

We're trying to prove an existential statement. The easiest way to do that is to just give concrete choices of the objects being sought out.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

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&= n.
\end{aligned}
$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

Check the appendix to this slide deck for more about who gets to choose values.

Time-Out for Announcements!

Working in Pairs

- Problem Set Zero is due this Monday at 1:00PM. It must be completed individually.
- After that, the remaining problem sets can be done individually or in pairs.
  - Each pair should make a single joint submission.
- We have advice about how to work effectively in pairs up on the course website – check the “Guide to Partners.”
- Want to work in a pair, but don’t know who to work with? Fill out ***this Google form*** and we’ll connect you with a partner on Monday.

Problem Set One

- Problem Set One goes out today. It’s due next Friday at 1:00PM.
    - Explore the language of set theory and better intuit how it works.
    - Learn more about the structure of mathematical proofs.
    - Write your first “freehand” proofs based on your experiences.
- As always, reach out if you have any questions!

Lecture 02

- Because our first lecture this quarter was on Wednesday, there’s one additional lecture we’ll need to cover before the start of next week.
- That lecture is prerecorded and is available on Canvas.
- It’s important to watch this lecture before Monday’s lecture and before starting Problem Set 1.
- Please feel free to post questions on EdStem!

Office Hours

- It is ***completely normal*** in this class to need to get help from time to time.
- Feel free to ask clarifying and conceptual questions on EdStem.
- Need more structured help? We have office hours! Feel free to stop on by.
  - Check out the online "Guide to Office Hours" for more information about how our office hours system works.
  - The OH calendar is available on the course website.
- Office hours start this Monday.

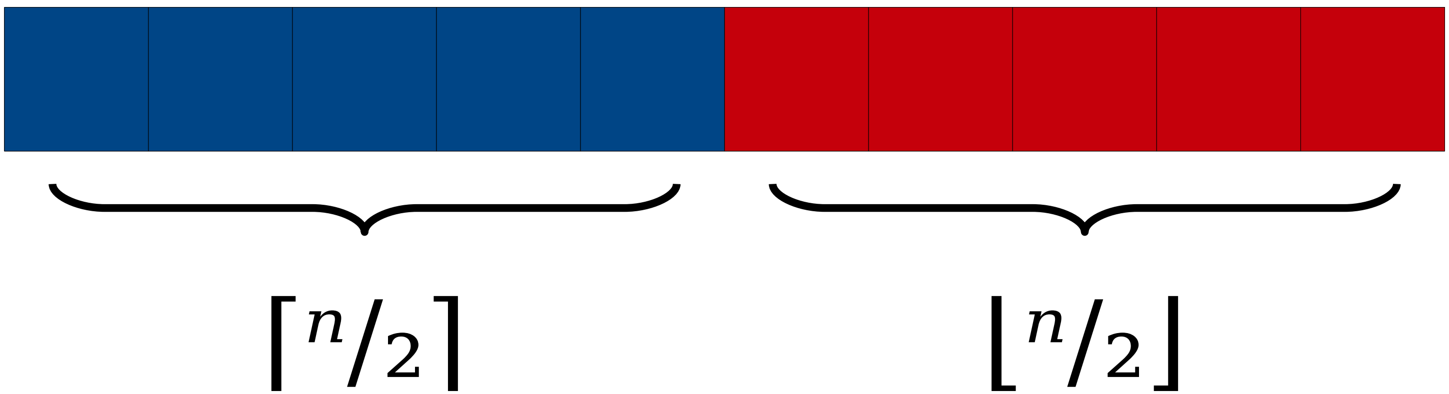
Back to CS103!

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

Floors and Ceilings

- The notation $\lceil x \rceil$ represents the ***ceiling*** of x, the smallest integer greater than or equal to x.
  - What is $\lceil 1 \rceil$? What's $\lceil 1.2 \rceil$? What's $\lceil -1.2 \rceil$?
  - ***Intuition:*** Start at x on the number line, then move to the right until you hit a tick mark.
- The notation $\lfloor x \rfloor$ represents is the ***floor*** of x, the largest integer less than or equal to x.
  - What is $\lfloor 1 \rfloor$? What's $\lfloor 1.2 \rfloor$? What's $\lfloor -1.2 \rfloor$?
  - ***Intuition:*** Start at x on the number line, then move to the left until you hit a tick mark.

Let's Draw Some Pictures!



$$n = 2k$$

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

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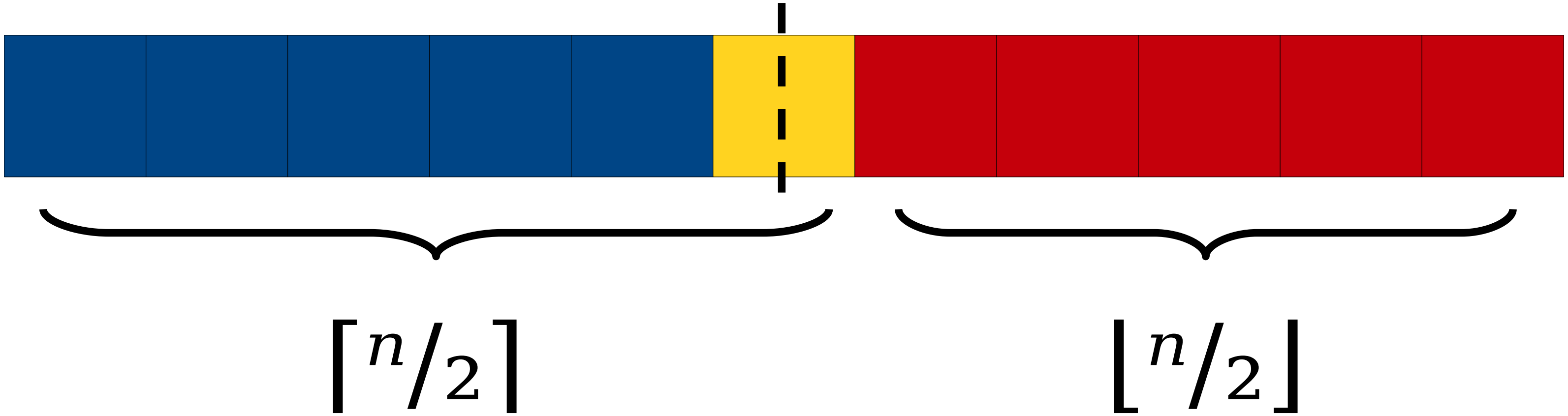

$$n = 2k + 1$$

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

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Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even.

Case 2: n is odd.

This is called a ***proof by cases*** (or ***proof by exhaustion***). We split apart into one or more cases and confirm that the result is indeed true in each of them.

(Think of it like an if/else or switch statement.)

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even. This means there is an integer k such that $n = 2k$. Some algebra then tells us that

$$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{2k}{2} \right\rceil + \left\lfloor \frac{2k}{2} \right\rfloor$$
$$= \lceil k \rceil + \lfloor k \rfloor$$
$$= 2k$$
$$= n\,.$$

Case 2: n is odd. Then there's an integer k where $n = 2k + 1$, and

$$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{2k+1}{2} \right\rceil + \left\lfloor \frac{2k+1}{2} \right\rfloor$$

At the end of a split into cases, it's a nice courtesy to explain to the reader what it was that you established in each case.

$$= n\,.$$

In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required.

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even. This means there is an integer k such that $n = 2k$. Some algebra then tells us that

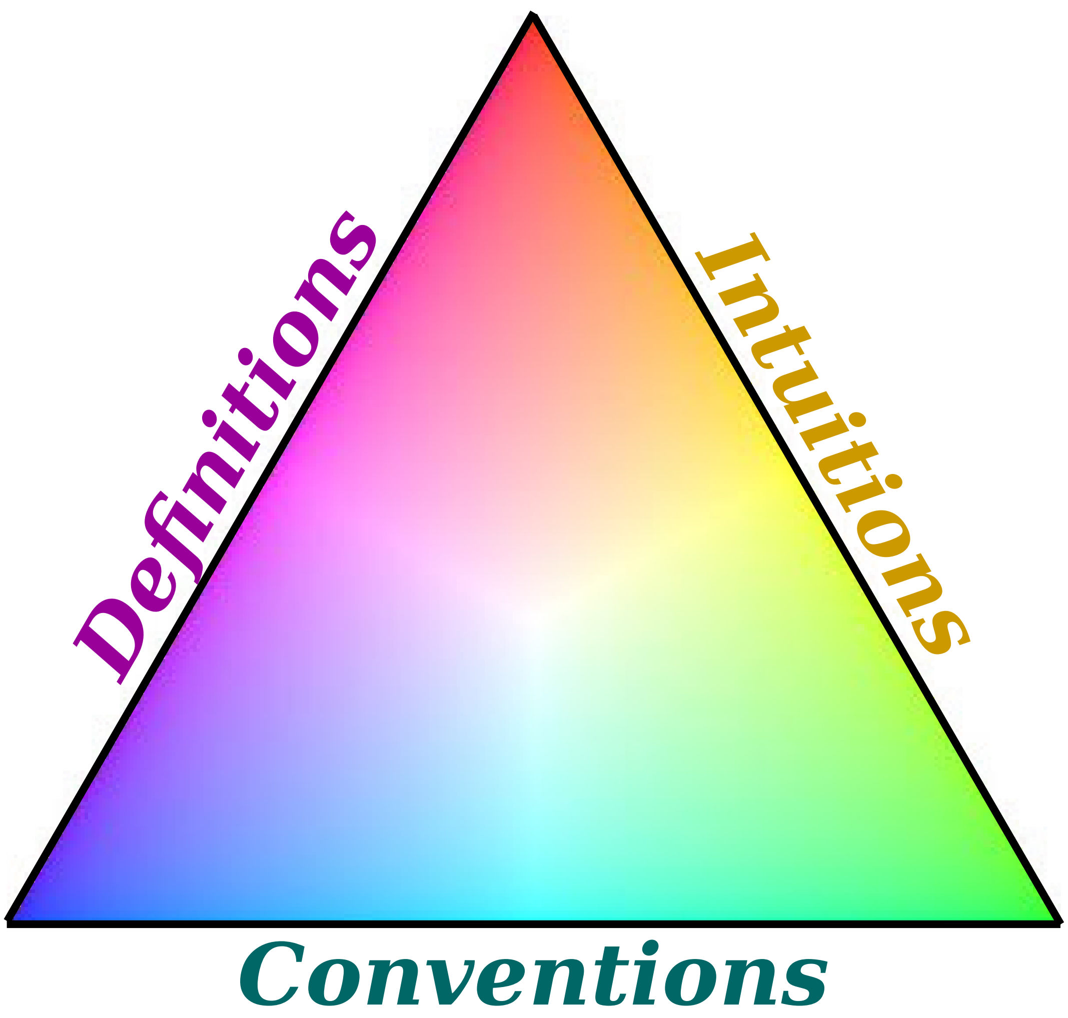
$$\begin{aligned} \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor &= \left\lceil \frac{2k}{2} \right\rceil + \left\lfloor \frac{2k}{2} \right\rfloor \\ &= \lceil k \rceil + \lfloor k \rfloor \\ &= 2k \\ &= n. \end{aligned}$$

Case 2: n is odd. Then there's an integer k where $n = 2k + 1$, and

$$\begin{aligned} \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor &= \left\lceil \frac{2k+1}{2} \right\rceil + \left\lfloor \frac{2k+1}{2} \right\rfloor \\ &= \left\lceil k + \frac{1}{2} \right\rceil + \left\lfloor k + \frac{1}{2} \right\rfloor \\ &= (k+1) + k \\ &= 2k+1 \\ &= n. \end{aligned}$$

In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required. ■

To Recap



Writing a good proof requires a blend of definitions, intuitions, and conventions.

An integer n is ***even*** if there is an integer k where $n = 2k$.

An integer n is ***odd*** if there is an integer k where $n = 2k+1$.

Definitions tell us what we need to do in a proof.
Many proofs directly reference these definitions.

Let's Draw Some Pictures!

Let's Do Some Math!

Let's Try Some Examples!

Building intuition for results requires creativity, trial, and error.

- Prove universal statements by making arbitrary choices.
- Prove existential statements by making concrete choices.
- Prove "If P, then Q" by assuming P and proving Q.
- Write in complete sentences.
- Number subformulas when referring to them.
- Summarize what was shown in proofs by cases.
- Articulate your start and end points.

Mathematical proofs have established conventions that increase rigor and readability.

Your Action Items

- ***Read "Guide to ∈ and ⊆," "Guide to Proofs," and "Guide to Partners."***
  - These will be very useful for PS1.
- ***Finish and submit Problem Set 0.***
  - Don't put this off until the last minute!
- ***Watch Lecture 02.***
  - Curl up near a warm fire and enjoy learning some more proof magic!
- ***Start Problem Set 1.***
  - At a minimum, read over the problems so you know what's being asked.

Next Time (Virtually!)

- ***Indirect Proofs***
  - How do you prove something without actually proving it?
- ***Mathematical Implications***
  - What exactly does "if P, then Q" mean?
- ***Proof by Contrapositive***
  - A helpful technique for proving implications.
- ***Proof by Contradiction***
  - Proving something is true by showing it can't be false.

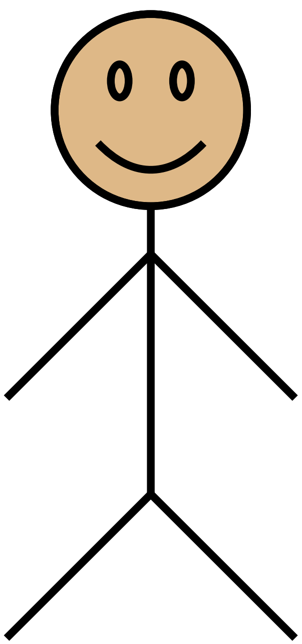
Appendix: ***Proofs as Dialogs***

Proofs as a Dialog

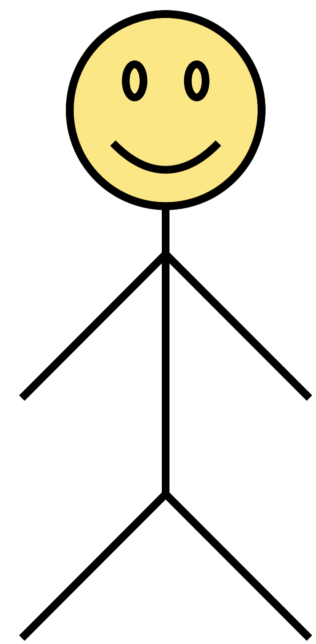
Let n be an arbitrary odd integer.

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

Now, let $z = k - 34$.



Proof Writer (You)



Proof Reader

Proofs as a Dialog

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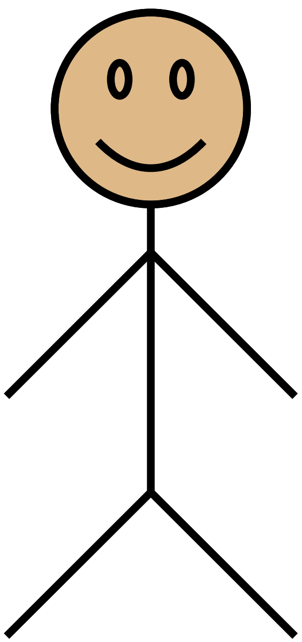
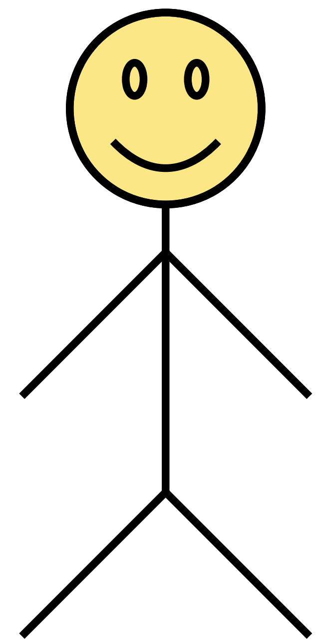


Proof Writer (You)

$n = 137$

Reader *Picks*



Proof Reader

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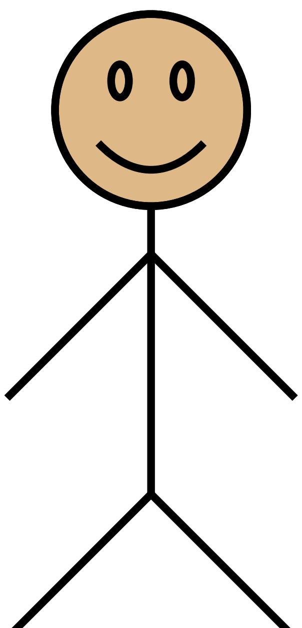


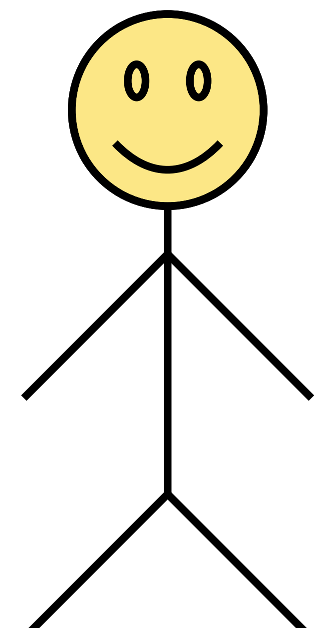
Proof Writer (You)

$k = 68$

Neither *Picks*

$n = 137$

Reader *Picks*



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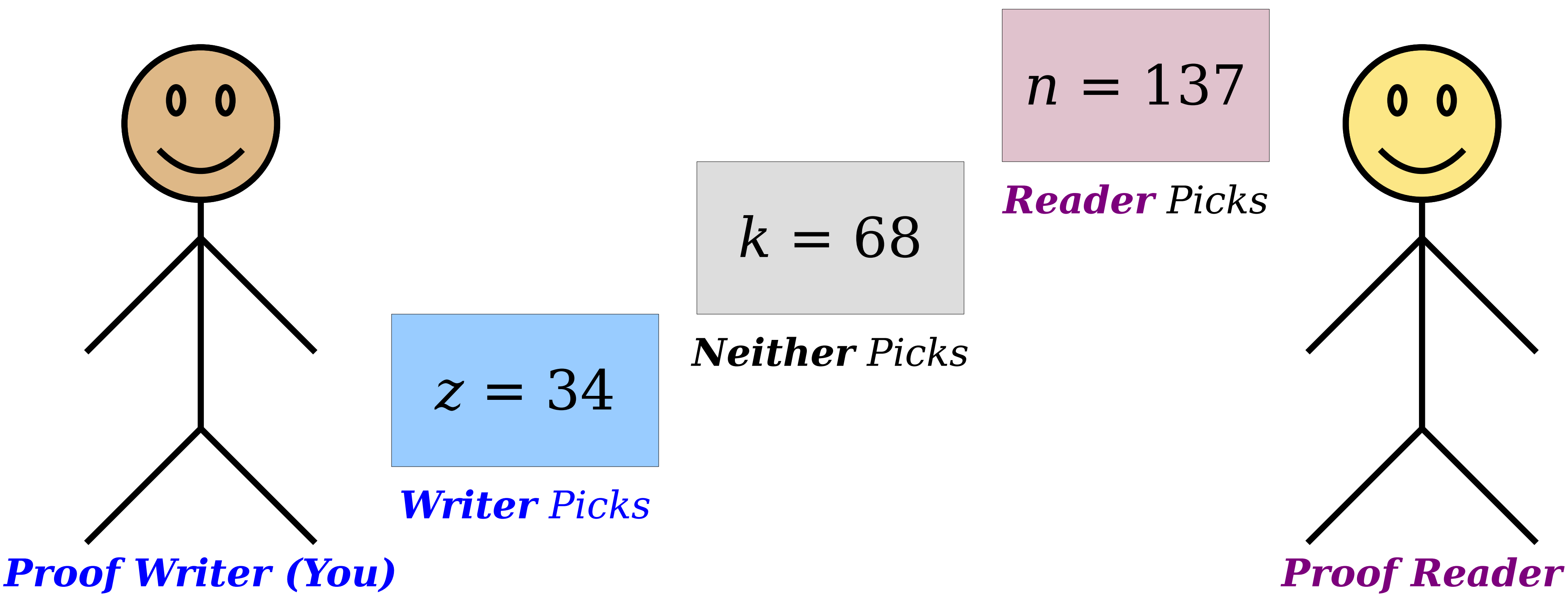

Proofs as a Dialog

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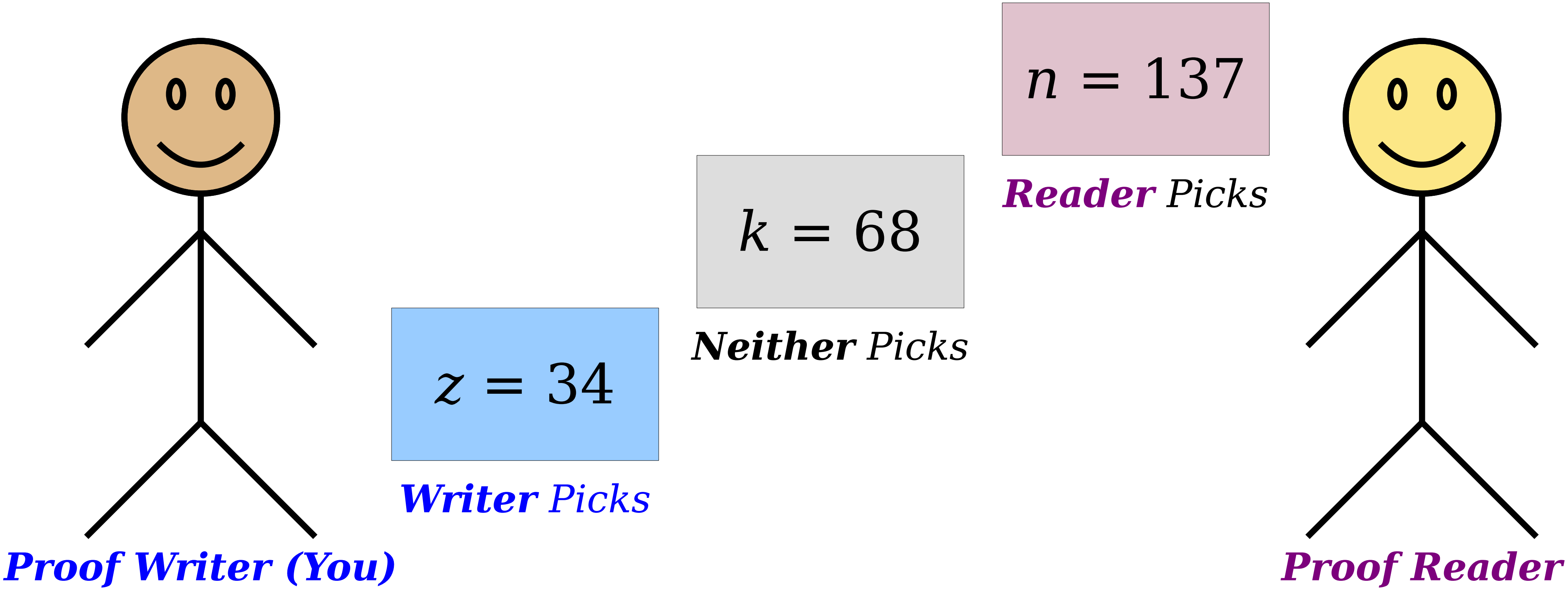
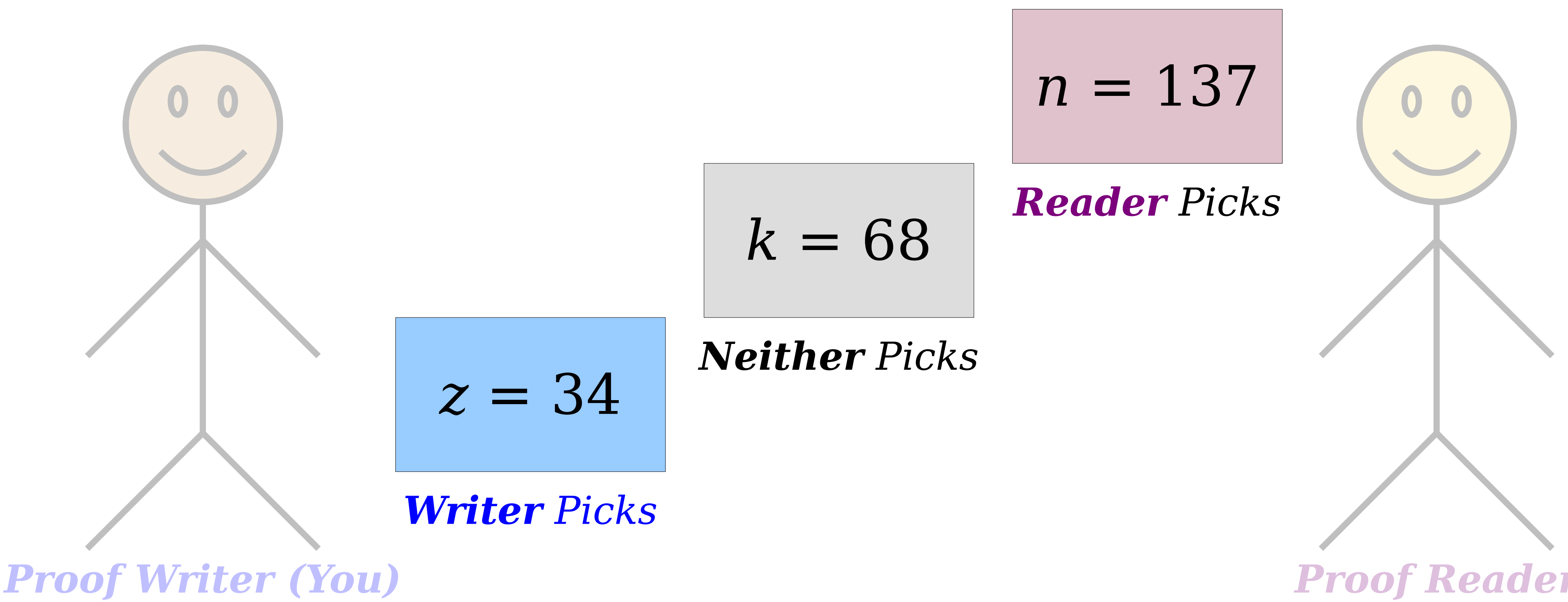


Each of these variables has a distinct, assigned value.

Each variable was either picked by the reader, picked by the writer, or has a value that can be determined from other variables.

Since … r

Now, let $z = k - 34$.



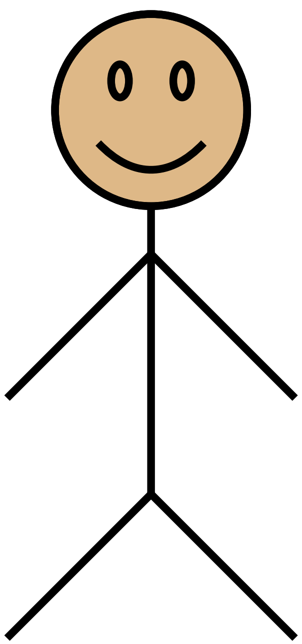
Who Owns What?

- The ***reader*** chooses and owns a value if you use wording like this:
  - Pick a natural number n.
  - Consider some $n \in \mathbb{N}$.
  - Fix a natural number n.
  - Let n be a natural number.
- The ***writer*** (you) chooses and owns a value if you use wording like this:
  - Let $r = n + 1$.
  - Pick $s = n$.
- ***Neither*** of you chooses a value if you use wording like this:
  - Since n is even, we know there is some $k \in \mathbb{Z}$ where $n = 2k$.
  - Because n is odd, there must be some integer k where $n = 2k + 1$.

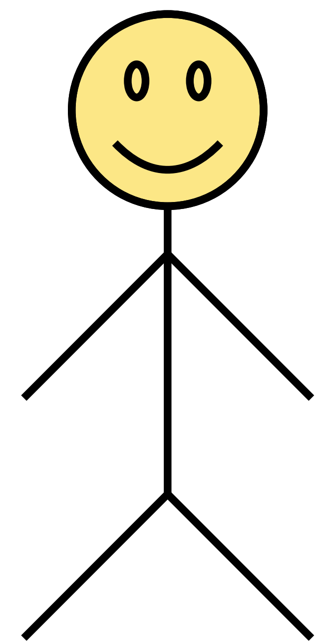
Proofs as a Dialog

Let x be an arbitrary even integer.

Then for any even x, we know that x+1 is odd.



Proof Writer (You)

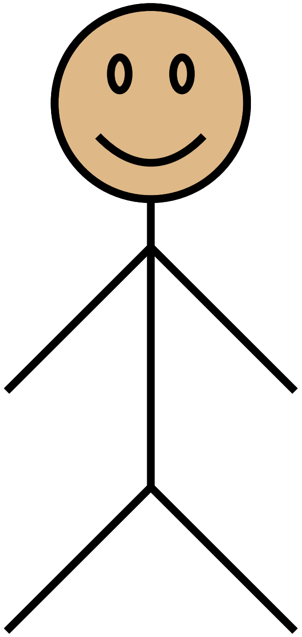


Proof Reader

Proofs as a Dialog

Let x be an arbitrary even integer.

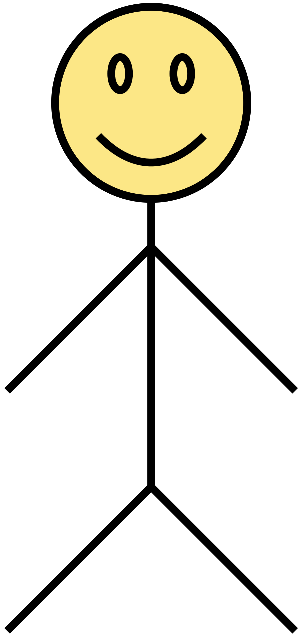
Then for any even x, we know that $x+1$ is odd.



Proof Writer (You)

$x = 242$

Reader *Picks*

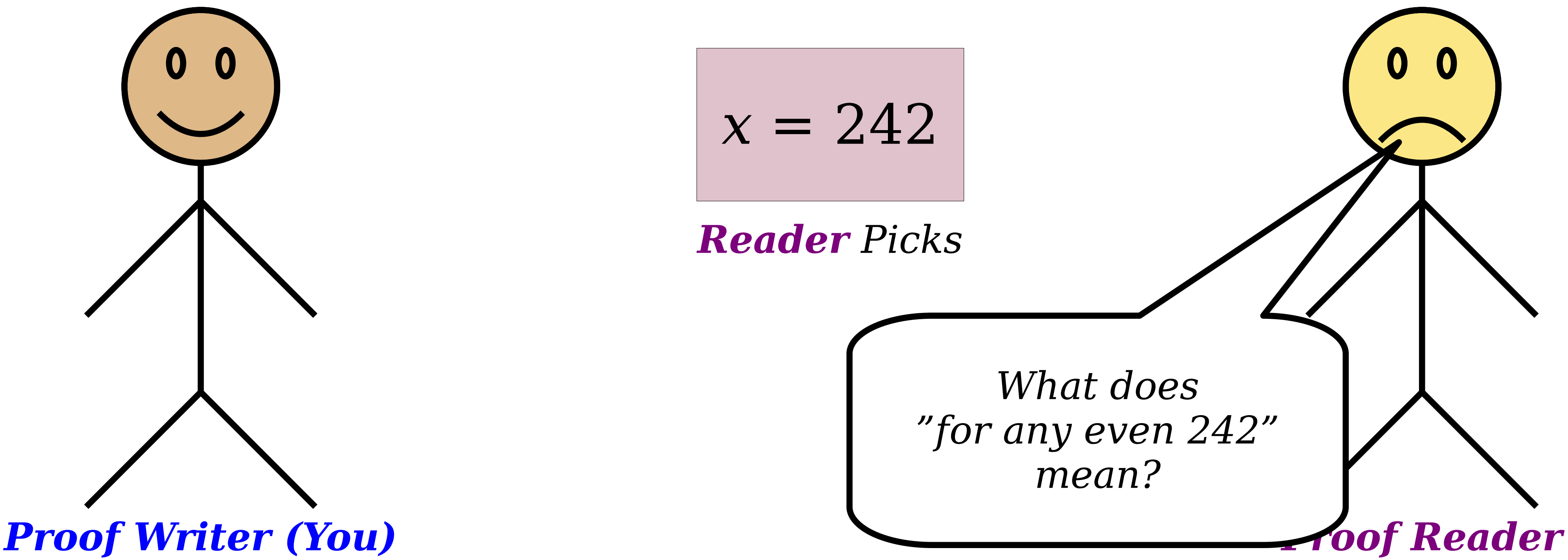


Proof Reader

Proofs as a Dialog

Let x be an arbitrary even integer.

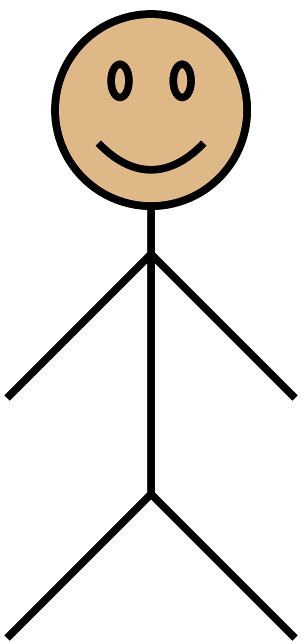
Then **for any even x**, we know that $x+1$ is odd.



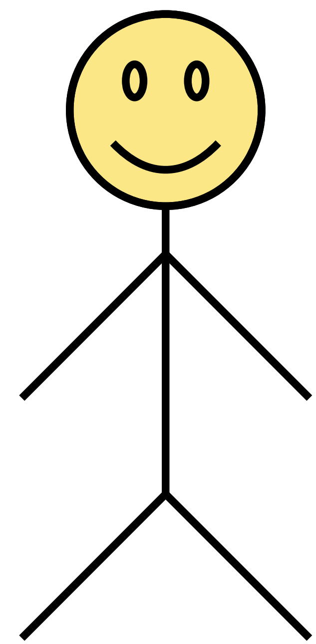
Proofs as a Dialog

Let x be an arbitrary even integer.

Since x is even, we know that $x+1$ is odd.



Proof Writer (You)



Proof Reader

Proofs as a Dialog

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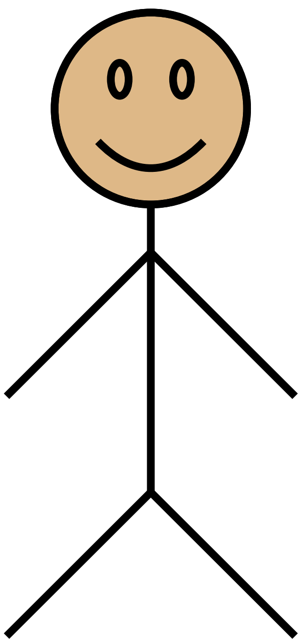
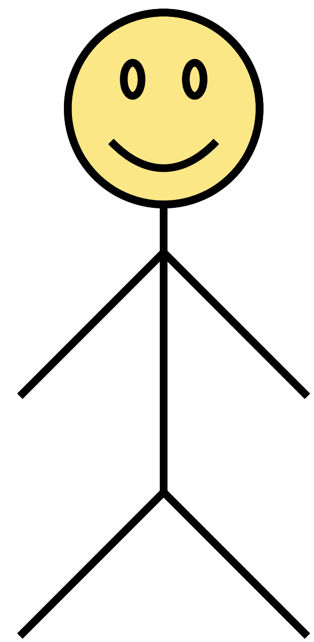


Proof Writer (You)

$x = 242$

Reader *Picks*



Proof Reader

Proofs as a Dialog

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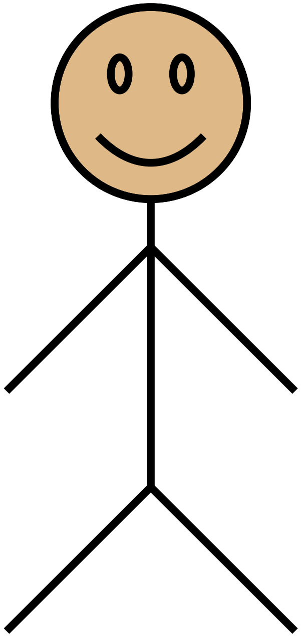
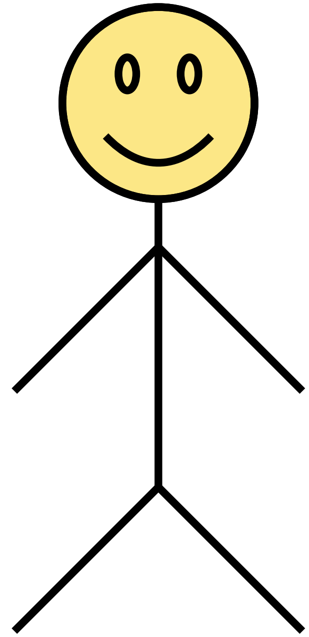


$x = 242$

***Reader** Picks*

Proof Writer (You)



Proof Reader

Every variable needs a value.

Avoid talking about "all x" or "every x" when manipulating something concrete.

To prove something is true for any choice of a value for x, let the reader pick x.

Once you've said something like

Let x be an integer.
Consider an arbitrary $x \in \mathbb{Z}$.
Pick any x.

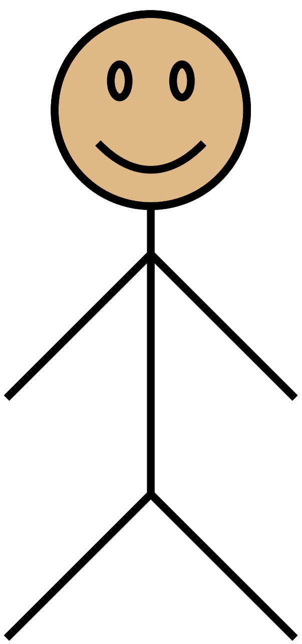
Do not say things like the following:

This means that ***for any*** $x \in \mathbb{Z}$ …
So ***for all*** $x \in \mathbb{Z}$ …

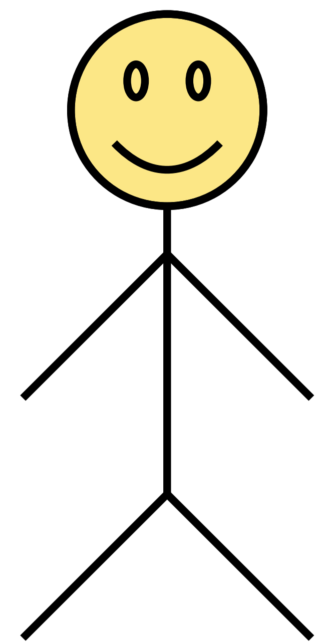
Proofs as a Dialog

Pick two integers m and n where $m+n$ is odd.

Let $n = 1$, which means that $m+1$ is odd.



Proof Writer (You)



Proof Reader

Proofs as a Dialog

Pick two integers m and n where $m+n$ is odd.

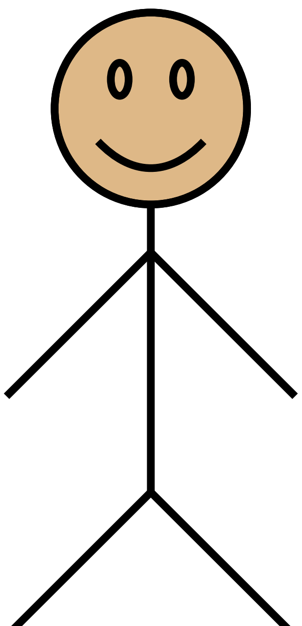
Let $n = 1$, which means that $m+1$ is odd.

$m = 103$

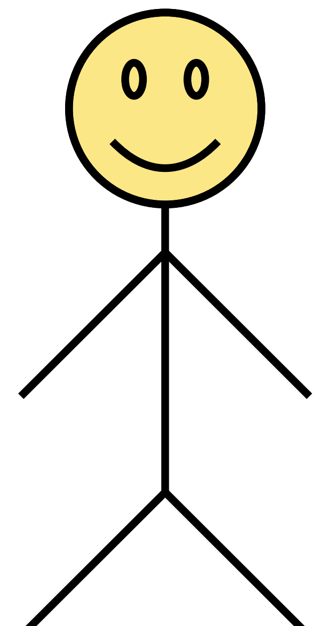
Reader *Picks*

$n = 166$

Reader *Picks*



Proof Writer (You)



Proof Reader

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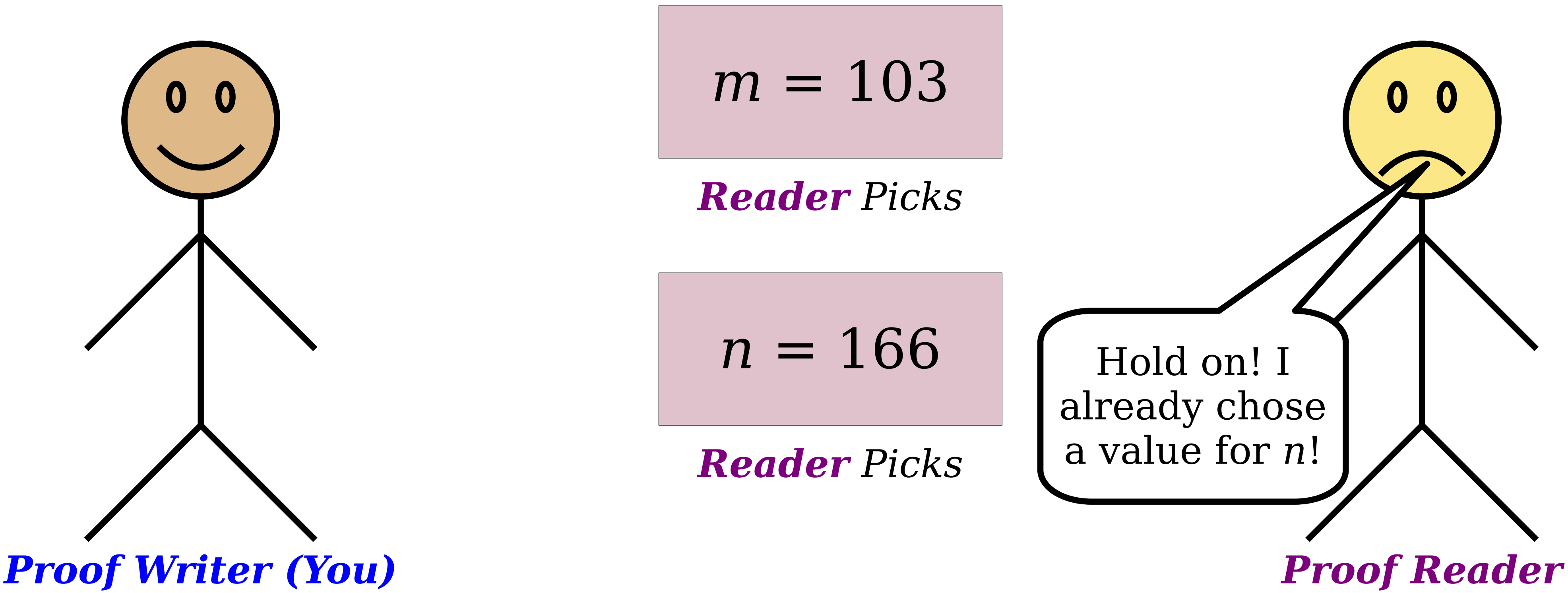
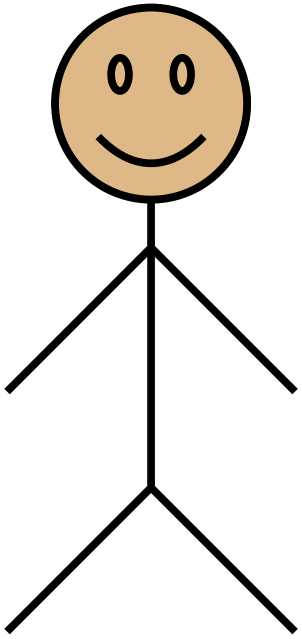


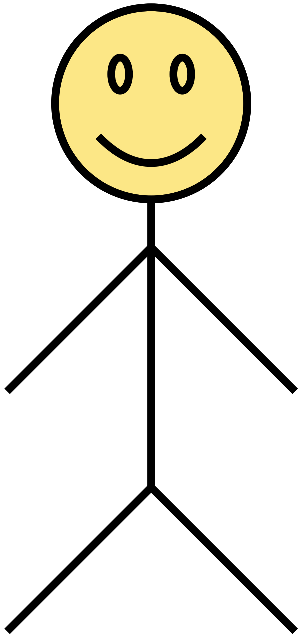
Proofs as a Dialog

Let $n = 1$.

Pick any integer m where $m+1$ is odd.



Proof Writer (You)



Proof Reader

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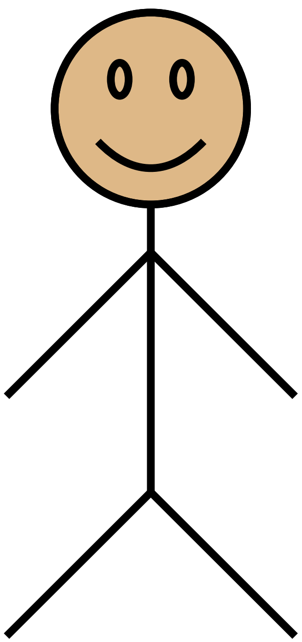
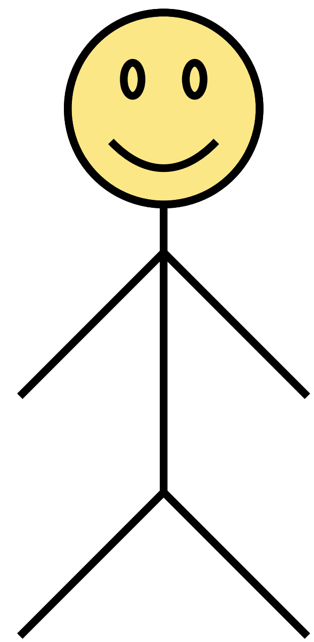


Proof Writer (You)

$n = 1$

Writer *Picks*



Proof Reader

Proofs as a Dialog

Let n = 1.

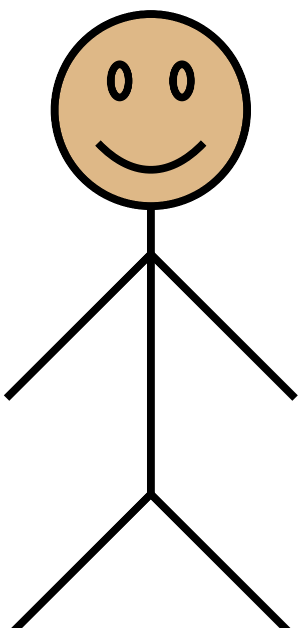
Pick any integer m where m+1 is odd.

$m = 166$

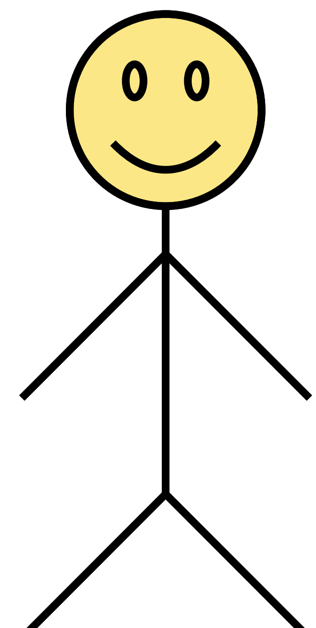
Reader *Picks*

$n = 1$

Writer *Picks*



Proof Writer (You)



Proof Reader

Proofs as a Dialog

Let $n = 1$.

Pick any integer m where $m+1$ is odd.

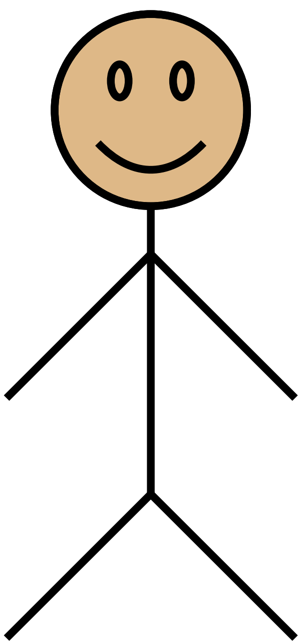
Do we even need n here?

$m = 166$

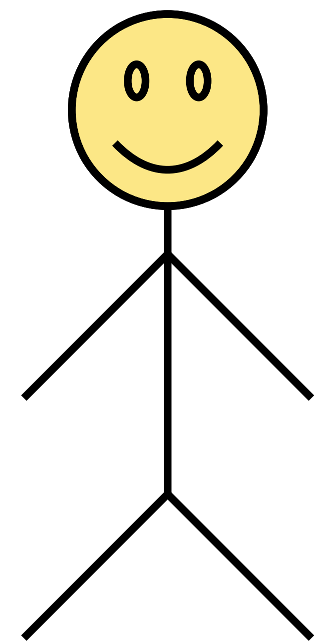
***Reader** Picks*

$n = 1$

***Writer** Picks*



Proof Writer (You)



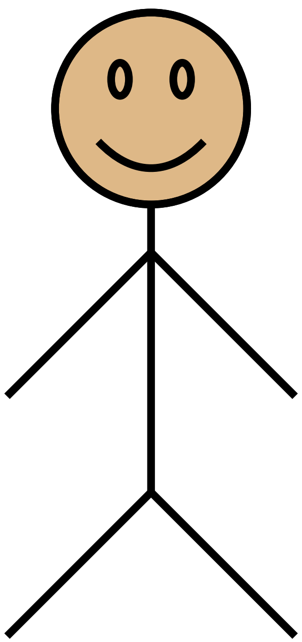
Proof Reader

Proofs as a Dialog

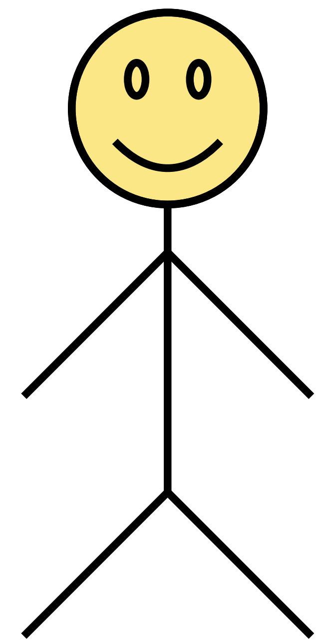
Pick any integer m where $m+1$ is odd.

$m = 166$

Reader *Picks*



Proof Writer (You)



Proof Reader

Be mindful of who owns what variable.

Don’t change something you don’t own.

You don’t always need to name things, especially if they already have a name.